Foundations of the golden ratio base

Jeffrey Quesnelle

April 3, 2015

Positional numeration systems have come to dominate mathematics, with the ubiquitous base-ten number system used nearly universally. In addition to base-ten, other bases such as base-two and base-sixteen have found widespread usage (for example in computer engineering). We review a particularly novel take on the positional numeration system: the golden ratio base, first introduced by George Bergman in 1957 [1], who was a 12 year old junior high student at the time[1]. We shall prove that the number system is correct, starting with basic properties of the golden ratio up to proofs of the existence and uniqueness of representations for certain classes of numbers, which rely on algebraic number theory (none of which were concretely shown in [1]). In addition we will introduce simpler algorithms for performing arithmetic in the system than were given in [1].

Background

Recall that the golden ratio is

$$\phi = \frac{1+\sqrt{5}}{2}.$$

A history of the golden ratio is beyond the scope of this paper. Instead, we will base the results of our report on a peculiar property of ϕ and its connection to the so-called Fibonacci numbers.

Proposition 1. *Let A be a recurrence relation with $A_n = A_{n-1} + A_{n-2}$. Then, $\lim\limits_{n\to 0} \frac{A_{n+1}}{A_n} = \phi$.*

Proof. We first note that

$$1+\frac{1}{\phi} = 1+\frac{1}{\frac{1+\sqrt{5}}{2}} = 1+\frac{2}{1+\sqrt{5}} = 1+\frac{2(\sqrt{5}-1)}{(1+\sqrt{5})(\sqrt{5}-1)} = 1+\frac{2(\sqrt{5}-1)}{4} = \frac{1+\sqrt{5}}{2} = \phi. \tag{1}$$

Let $R_n = \frac{A_{n+1}}{A_n}$. Expanding R_n,

$$R_n = \frac{A_{n+1}}{A_n} = \frac{A_n + A_{n-1}}{A_n} = 1 + \frac{1}{R_{n-1}}. \tag{2}$$

If we let $L = \lim\limits_{n\to\infty} R_n$, then clearly $\lim\limits_{n\to\infty} R_{n+1} = L$ as well. Then by (2),

$$L = \lim_{n\to\infty} R_{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{R_n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{\lim\limits_{n\to\infty} R_n}\right) = 1+\frac{1}{L}.$$

[1] Bergman's academic institution in [1] is listed as "Jr. High School 246, Brooklyn, N.Y."

Already by (1) we can see that $L = \phi$ works. Going full circle, we can compute L directly: $L^2 = L\left(1 + \frac{1}{L}\right) = L + 1 \Rightarrow L^2 - L - 1 = 0$. Using the quadratic formula, $L = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1+\sqrt{5}}{2} = \phi$. $\square$

Proposition 2. *Let ϕ be the golden ratio and F_n be the Fibonacci sequence. Then, $\phi^n = \phi^{n-1} + \phi^{n-2} = F_n\phi + F_{n-1}$.*

Proof. The simplest method for proving $\phi^n = \phi^{n-1} + \phi^{n-2}$ is by induction. We directly compute for the base case of $n = 2$.

$$\phi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2} + 1 = \phi^1 + \phi^0.$$

Now, assuming that the proposition holds for all natural numbers greater than 2 and less than n, we need to show that $\phi^n = \phi^{n-1} + \phi^{n-2}$. Since $\phi^n = \phi\phi^{n-1}$, by the inductive hypothesis: $\phi^n = \phi\phi^{n-1} = \phi\left(\phi^{n-2} + \phi^{n-3}\right) = \phi^{n-1} + \phi^{n-2}$.

Given this result, we can show that $\phi^n = F_n\phi + F_{n-1}$. Recall that the Fibonacci sequence is generated by the recurrence relation in Proposition 1 with initial conditions $F_0 = 0$ and $F_1 = 1$. (The first few numbers therefore are 0, 1, 1, 2, 3, 5, 8, ...). For simplicities stake we will again use the base case of 2; $\phi^2 = (1)\phi + 1 = F_2\phi + F_1$. If this holds true for all natural numbers up to n, then $\phi^n = \phi^{n-1} + \phi^{n-2} = (F_{n-1}\phi + F_{n-2}) + (F_{n-2}\phi + F_{n-3}) = (F_{n-1} + F_{n-2})\phi + (F_{n-2} + F_{n-3}) = F_n\phi + F_{n-1}$. $\square$

The golden ratio base

The golden ratio base is a positional base system using the digits 0 and 1. The digits represent powers of ϕ; non-negative on the left side of the decimal point and negative on the right. For example[2], $101.01_\phi = \phi^2 + \phi^0 + \phi^{-2} = (1\phi + 1) + (1) + (-1\phi + 2) = 4$. One of the properties of base-ϕ is that numbers have several representations: $100.1111_\phi = \phi^2 + \phi^{-1} + \phi^{-2} + \phi^{-3} + \phi^{-4} = (1\phi + 1) + (1\phi - 1) + (-1\phi + 2) + (2\phi - 3) + (-3\phi + 5) = 4$. Thus, $101.01_\phi = 100.1111_\phi = 4$, which is not immediately obvious.

By Proposition 2, $\phi^n = \phi^{n-1} + \phi^{n-2}$, or in terms of base-$\phi$: $100_\phi = 11_\phi$. So, whenever we encounter two consecutive 1's in a base-ϕ representation there is an equivalent representation that uses one less 1: $100.1111_\phi = 101.0011_\phi = 101.01_\phi = 4$. We can continually repeat this process until there are no adjacent 1's.

Definition 3. A number in base-ϕ is in *standard form* if it contains no consecutive 1's.

Existence and uniqueness

Can all positive integers be uniquely represented in standard form? To see that this is true [2] we will study $\mathbb{Z}[\phi] = \{a + b\phi \mid a, b \in \mathbb{Z}\}$, the quadratic integer subring of the quadratic field $\mathbb{Q}(\sqrt{5})$ equipped with the usual operations of multiplication and addition (analogous to complex arithmetic). Since $\mathbb{Z}[\phi] \subseteq \mathbb{R}$, we will use the natural ordering of $\mathbb{R}$ for $<$ and $\leq$. We define the conjugate of $\alpha \in \mathbb{Z}[\phi]$ as $\overline{\alpha} = a - b/\phi$ and the norm $N(\alpha) = \alpha\overline{\alpha} = a^2 + ab - b^2$. Of additional note is that $\overline{\alpha} = \alpha - b\sqrt{5}$ and so

$$N(\alpha) = \alpha\overline{\alpha} = \alpha(a - b/\phi) = \alpha(\alpha - b\sqrt{5}). \tag{3}$$

[2]Such a base requires using the "extended" Fibonacci sequence where $f_{-n} = (-1)^{n+1}f_n$

Our norm satisfies the property $N(\alpha\beta) = N(\alpha)N(\beta)$ for $\alpha, \beta \in \mathbb{Z}[\phi]$. Additionally, $N(\phi) = N(\phi^{-1}) = -1$ and thus

$$|N(\phi^k)| = 1 \text{ for all } k \in \mathbb{Z}. \tag{4}$$

Lastly, $N(\alpha) = 0 \iff \alpha = 0$.

Lemma 4. *If $\alpha = a + b\phi, \alpha \in \mathbb{Z}[\phi]$ with $1 \leq \alpha < \sqrt{5}$, then $|N(\alpha)| > |N(\alpha - 1)|$.*

Proof. Let $\gamma = c + d\phi$, $\gamma \in \mathbb{Z}[\phi]$ with $0 \leq \gamma < \sqrt{5}$. Using (3) we have

$$|N(\gamma)| = \gamma|c - d/\phi| = \gamma|\gamma - d\sqrt{5}| = \begin{cases} \gamma(d\sqrt{5} - \gamma) & \text{if } d \geq 1, \\ \gamma(\gamma - d\sqrt{5}) & \text{if } d < 1. \end{cases} \tag{5}$$

Since $1 \leq \alpha < \sqrt{5}$ we can use (5) to calculate $N(\alpha)$ and $N(\alpha - 1)$. If $b \geq 1$, then $b\sqrt{5} \geq 2\alpha - 1$ and the first case of (5) applies: $|N(\alpha)| = \alpha(\alpha - b\sqrt{5}) > (\alpha - 1)(b\sqrt{5} - (\alpha - 1)) = |N(\alpha - 1)|$. Otherwise, when $b < 1$ and $b\sqrt{5} < 2\alpha - 1$ we have $|N(\alpha)| = \alpha(\alpha - b\sqrt{5}) > (\alpha - 1)\left((\alpha - 1) - b\sqrt{5}\right) = |N(\alpha - 1)|$. Either way, $|N(\alpha)| > |N(\alpha - 1)|$. □

Theorem 5. *For every positive integer n, there is a corresponding finite sequence of distinct integers $k_1, k_2, \cdots, k_m$ such that $n = \phi^{k_1} + \phi^{k_2} + \cdots + \phi^{k_m}$ with $k_i - k_{i+1} \geq 2$ for all $1 \leq i < m$, i.e. n has a finite standard form base-ϕ representation.*

Proof. Let σ be a positive number in $\mathbb{Z}[\phi]$. Then, pick $k \in \mathbb{Z}$ such that $\phi^k \leq \sigma < \phi^{k+1}$ and let $\alpha = \phi^{-k}\sigma$. Since $1 \leq \alpha < \phi < \sqrt{5}$, by (4) we know that $|N(\alpha)| = |N(\phi^{-k}\sigma)| = |N(\phi^{-k})N(\sigma)| = |N(\sigma)|$. Furthermore, since $\alpha = \phi^{-k}\sigma$, $\alpha - 1 = \phi^{-k}\sigma - 1 = \phi^k(\phi^{-k}\sigma - 1) = \sigma - \phi^k$ and so $|N(\alpha - 1)| = |N(\sigma - \phi^k)|$. By Lemma 4 we know that $|N(\alpha)| > |N(\alpha - 1)|$, which gives us the result that $|N(\sigma)| > |N(\sigma - \phi^k)|$. If $\sigma > \phi^k$ we can repeat this process for $\sigma' = \sigma - \phi^k$. We know that each successive ϕ^k choice reduces the norm: $|N(\sigma)| > |N(\sigma - \phi^{k_1})| > |N(\sigma - \phi^{k_1} - \phi^{k_2})| > \cdots \geq 0$. Since $|N(\alpha)|$ is a positive integer this process eventually stops, and $|N(\sigma - \phi^{k_1} - \phi^{k_2} - \cdots - \phi^{k_m})| = 0 \Rightarrow \sigma - \phi^{k_1} - \phi^{k_2} - \cdots - \phi^{k_m} = 0 \Rightarrow \sigma = \phi^{k_1} + \phi^{k_2} + \cdots + \phi^{k_m}$.

Each of the integers $k_1, k_2, \cdots, k_m$ are distinct. In addition, for each k we have $\phi^k \leq \sigma < \phi^{k+1} \Rightarrow 0 \leq \sigma - \phi^k < \phi^{k-1}$ and thus for all k_i, $k_i - k_{i+1} \geq 2$, so our base-ϕ representation is in standard form. Because σ is an arbitrary element in $\mathbb{Z}[\phi]$ and $\mathbb{Z}^+ \subseteq \mathbb{Z}[\phi]$, we conclude that each positive integer has a finite standard form representation in base-ϕ. □

One may notice the use of the word finite in Theorem 5. Is this necessary? In fact, in every base-n system all terminating representations have an alternate non-terminating and recurring representation [3]. Analogous to the infamous $0.999\ldots = 1$ result in base-10, in base-ϕ we have $0.101010\ldots_\phi = 1_\phi$.

Proposition 6. *For every non-zero number n in base-ϕ the final 1 can be replaced with* $010101\ldots$.

Proof. Since $n \neq 0$, n contains at least one 1 in base-ϕ. Let ϕ^k correspond to the position of the final 1 in n. Consider

$$\sum_{i=0}^{\infty} \phi^{-2i+(k-1)} = 101010\ldots_\phi \text{ starting at } k-1 \text{ (position to the right of the last 1).} \tag{6}$$

Since (6) is an infinite geometric series with common ratio $\phi^{-2} \approx 0.3819$ and start term ϕ^{k-1}, we have

$$\sum_{i=0}^{\infty} \phi^{-2i+(k-1)} = \frac{\phi^{k-1}}{1-\phi^{-2}} = \frac{\phi^2}{\phi^2}\frac{\phi^{k-1}}{1-\phi^{-2}} = \frac{\phi^{k+1}}{\phi^2 - 1} = \frac{\phi^{k+1}}{F_2\phi + F_1 - 1} = \frac{\phi^{k+1}}{\phi} = \phi^k. \tag{7}$$

By (7), Adding $101010\ldots_\phi$ after the last (at position k) 1 of n has the effect of increasing n by ϕ^k. So, by removing ϕ^k (putting a 0 at k), we have $n - \phi^k + \phi^k = n$, which was to be shown. □

Interestingly enough the repeating form of any integer is also in standard form (assuming the original integer was in standard form). This is why Theorem 5 restricted uniqueness to *finite* standard forms.

Proposition 7. *All finite standard base-ϕ representations of the positive integers are unique.*

Proof. Let $\alpha, \beta \in \mathbb{Z}^+$ with $\alpha = \beta$ but having different standard base-ϕ representations. Let $k_1^\alpha, k_2^\alpha, \ldots, k_m^\alpha$ be the powers of ϕ for α and $k_1^\beta, k_2^\beta, \ldots, k_n^\beta$ be the powers of ϕ for β with both being ordered highest to lowest (i.e. $k_1^\beta > k_2^\beta > \cdots$).

For our first case, without loss of generality assume $k_1^\alpha > k_1^\beta$. It is worth noting that $\phi^z > 0$ for all $z \in \mathbb{Z}$, which tells us that removing any power of ϕ from a positive integer will make it smaller (and adding a power will make it larger). Since β is in standard from it contains no consecutive powers of ϕ. What is the largest possible standard form number whose highest power of ϕ is k_1^β? By Proposition 6 we know that the infinite standard form $101010_\phi\ldots$ starting at k is equal[3] to ϕ^{k+1}. But β has a *finite* standard form, so $\beta < \phi^{k_1^\beta+1}$. Since, $k_1^\alpha > k_1^\beta$ we have that $\beta < \phi^{k_1^\beta+1} \leq \phi^{k_1^\alpha} \leq \alpha$, a contradiction.

If $k_1^\alpha = k_1^\beta$ then there must be some $j < \max(m,n)$ where $k_j^\alpha \neq k_j^\beta$ i.e. the two base-ϕ representations are different so they must differ by at least one digit. We can repeat the above process with k_j^β replacing k_1^β resulting in the conclusion that

$$\beta - \sum_{i=1}^{j-1} \phi^{k_i^\beta} < \alpha - \sum_{i=1}^{j-1} \phi^{k_i^\alpha}$$

which again leads us to $\beta < \alpha$, a contradiction. □

Corollary 8. *Let n be a number with a finite base-ϕ representation in standard form whose highest digit is at place k. Then, $n < \phi^{k+1}$.*

Arithmetic in base-ϕ

To make our arithmetic methods more intuitive, we shall cheat slightly and create a new "extended" base-ϕ system that allows all of the integers as digits, not just 0 and 1. For example, $210.01_\phi = 2\cdot\phi^2 + \phi + \phi^{-2}$. Any number with a finite extended base-ϕ representation can be converted to a finite regular base-ϕ representation. Rather than provide a formal proof we shall simply illustrate the process for performing the conversion, with the hope that its correctness is obvious. We rely on two new identities; firstly, $2\cdot\phi^k = \phi^{k+1} + \phi^{k-2}$ which can be verified directly using Proposition 2:

$$\begin{aligned}
\phi^{k+1} + \phi^{k-2} &= \phi^k + \phi^{k-1} + \phi^{k-2} \\
&= \phi^k + \phi^k \\
&= 2\cdot\phi^k.
\end{aligned} \tag{8}$$

[3](7) used starting point $k-1$

The second identity is $-\phi^k = -\phi^{k+1} + \phi^{k-1}$. Similarly,

$$\begin{aligned} -\phi^k &= -\phi^k \\ \phi^{k+1} - \phi^k &= \phi^{k+1} - \phi^k \\ \phi^k + \phi^{k-1} - \phi^k &= \phi^{k+1} - \phi^k \\ -\phi^{k+1} + \phi^{k-1} &= -\phi^k \end{aligned} \tag{9}$$

The process for converting a positive number in extended base-ϕ to standard base-ϕ is as follows:

1. For any digit non-0,1 digit c at place k in the base-ϕ representation corresponding to $c \cdot \phi^k$, factor c to unique positive integers m, n, s: $c = s(2m + n)$ with $n \in \{0, 1\}$ and $s \in \{-1, 1\}$. This follows directly from the division algorithm, with divisor 2, extended to allow negatives. So, for $-5 \cdot \phi^k$ we would have $-1(2 \cdot 2 + 1)\phi^k$.

2. Add m to the digits $k + 1$ and $k - 2$. This follows from (8). Change the k digit to sn. This is the remainder of removing all the even factors from c. Repeat until all digits are either 1, 0, or -1.

3. For any digit k with the value -1, add -1 to digit $k + 1$ and add 1 to digit $k - 1$. Change digit k to 0. This follows from (9). If any digits are greater than 1, perform step 2. Repeat until no negatives remain, indicating the number is positive, or until a lone negative remains in the highest occupied digit, indicating the number is negative (which is not possible since the number is positive).

4. Convert the representation (which now contains only 0,1 digits) to standard form using the identity $11_\phi = 100_\phi$.

The above process works in two parts, first reducing all digits greater than 1 (or less than -1) by adding values to the right and left of the digit to maintain equality. For example, $50_\phi \Rightarrow 210.2_\phi \Rightarrow 1011.2_\phi \Rightarrow 1012.001_\phi \Rightarrow 1020.011_\phi \Rightarrow 1100.111_\phi$. Since this contains no negatives, step 3 can be skipped and we can directly convert to standard form: $1100.111_\phi \Rightarrow 10000.111_\phi \Rightarrow 10001.001_\phi$. For each digit we reduce we increase "away" from the digit in question; because the number has a finite extended base-ϕ representation this process will eventually terminate. This first part is analogous to "carrying" in standard addition, with the exception that we must carry to two different places as opposed simply directly to left as in base-10.

The second part of the process involves canceling out negatives. At this point all digits are either 1, 0, or -1. Each -1 digit can be "shifted over" one by increasing the digit to the right. If any digits to the left of this are 1, the negative will eventually "cancel out". If all of the digits to the left are 0 (assuming we've shifted all other negatives into this digit), then the number must be negative by Corollary 8. However, we've restricted ourselves to non-negative numbers, therefore all negatives will cancel out. This process is analagous to "borrowing" in standard subtraction, with the exception that we must borrow from two places. We now have a number in base-ϕ which can be trivially reduced to standard form.

Although the above process may not seem like a simplification, it makes the actual arithmetic operations trivial. For more complicated algorithms that do not rely on this method, see [1].

Addition

Simply add the digits place-wise, and then convert back to standard form.

1 0 1 0 . 1	$2\sqrt{5}+2$
+ 1 0 0 . 1 0 1	$2\sqrt{5}-1$
1 1 1 0 . 2 0 1	$\rightarrow$ *convert to standard form*
1 0 0 1 0 . 2 0 1	$\cdots$
1 0 0 1 1 . 0 0 2	$\cdots$
1 0 1 0 0 . 0 1 0 0 1	$4\sqrt{5}+1$

Subtraction

Again, simply subtract place-wise and convert back to standard form.

1 0 1 0 . 1	$2\sqrt{5}+2$
- 1 0 0 . 1 0 1	$2\sqrt{5}-1$
1 -1 1 0 . 0 0 -1	$\rightarrow$ *convert to standard form*
1 -1 1 0 . 0 -1 0 1	$\cdots$
2 0 . 0 -1 0 1	$\cdots$
2 0 . -1 0 1 1	$\cdots$
1 0 0 . 0 1	3

Multiplication

Multiplication is the same as in base-ten with no carry; we ignore the decimal point and re-insert it at sum of the number of decimal places.

1 0 1 0 . 1	$2\sqrt{5}+2$
× 1 0 0 . 1 0 1	$2\sqrt{5}-1$
1 0 1 0 1	*-3 digit*
1 0 1 0 1	*-1 digit*
+ 1 0 1 0 1	*2 digit*
1 0 1 1 1 2 0 2 0 1	$\rightarrow$ *insert decimal point*
1 0 1 1 1 2 . 0 2 0 1	$\rightarrow$ *convert to standard form*
1 1 0 0 2 0 . 0 3 0 1	$\cdots$
1 0 0 0 1 0 0 . 2 1 0 2	$\cdots$
1 0 0 0 1 0 1 . 0 1 2 0 0 1	$\cdots$
1 0 0 0 1 0 1 . 0 2 0 0 1 1	$\cdots$
1 0 0 0 1 0 1 . 1 0 0 2	$\cdots$
1 0 0 0 1 0 1 . 1 0 1 0 0 1	$\cdots$
1 0 0 0 1 1 0 . 0 0 1 0 0 1	$\cdots$
1 0 0 1 0 0 0 . 0 0 1 0 0 1	$2\sqrt{5}+18$

Division

Long division can also be performed, although one most be careful of decimal places.

1 0 0 . 0 1) $\overline{1\ 0\ 0\ 1\ 0\ .\ 0\ 1\ 0\ 1}$	$9 \div 3$
	$\rightarrow$ *move decimal place over by 2*
1 0 0 . 0 1	3
1 0 0 0 1) $\overline{1\ 0\ 0\ 1\ 0\ 0\ 1\ .\ 0\ 1}$	$\rightarrow$ *put 1 in 2's place*
- 1 0 0 0 1	1×10001
1 -1	$\rightarrow$ *convert to standard form*
1	$\rightarrow$ *bring down 0101*
1 1 0 1	$\rightarrow$ *convert to standard form*
1 0 0 0 1	$\rightarrow$ *put 1 in -2 place*
- 1 0 0 0 1	1×10001
0	*no remainder*

Rationals and irrationals

We now know that all positive integers have a finite base-ϕ representation. What about the rationals? Much like the familiar base-ten system, all rationals have a finite or repeating base-ϕ representation. To show this, we will prove something a little stronger.

Theorem 9. *Let q be a number. Then, q has a finite or infinitely recurring base-ϕ representation if and only if $q \in \mathbb{Q}(\sqrt{5})$.*

Proof. ($\Rightarrow$) If q has a finite base-ϕ representation then $q = \phi^{k_1} + \phi^{k_2} + \cdots + \phi^{k_n}$ for some $k_1, k_2, \ldots, k_n \in \mathbb{Z}$. Since $\phi \in \mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{5})$ is closed under addition and multiplication (by virtue of being a field), $q \in \mathbb{Q}(\sqrt{5})$. If q is infinitely recurring, then there must be some m with $k_m < 0$ where the recurrence first starts. If n is the location of the last element before the recurrence starts a second time, then, $l = k_m - k_n + 1$ is the length of the recurrence. Let $k_1 > k_2 > \cdots$. Then,

$$q = \phi^{k_1} + \phi^{k_2} + \cdots + \phi^{k_{m-1}} + \sum_{i=0}^{\infty} \left(\phi^{-il+k_m} + \phi^{-il+k_{m+1}} + \cdots \ \phi^{-il+k_n} \right). \tag{10}$$

We can distribute the summation to each individual term. For each term j with $m \leq j \leq n$, we have

$$\sum_{i=0}^{\infty} \phi^{-il+k_j}.$$

This is a geometric series with common ratio ϕ^{-l} and start term ϕ^{k_j}. Since $l > 0$ and $0 < \phi^a < 1$ for all $a < 0$, this series converges. So,

$$\sum_{i=0}^{\infty} \phi^{-il+k_j} = \frac{\phi^{k_j}}{1 - \phi^{-l}} = \frac{\phi^l}{\phi^l} \frac{\phi^{k_j}}{1 - \phi^{-l}} = \frac{\phi^{l+k_j}}{\phi^l - 1} \tag{11}$$

The denominator of (11) is problematic. However, we can expand ϕ^l via repeated applications of the identity $\phi^n = \phi^{n-1} + \phi^{n-2}$. If l is even we have $\phi^l = \phi^{l-1} + \cdots + \phi^7 + \phi^5 + \phi^3 + \phi^2$, i.e. all odd powers less than l as well as ϕ^2. If l is odd we have $\phi^l = \phi^{l-1} + \cdots + \phi^6 + \phi^4 + \phi^2 + \phi$, i.e. all even powers less than l as well as ϕ. In either case this contains ϕ^2. But, $\phi^2 = \phi + 1$, so in both cases we can cancel out the -1 in the denominator. Let $u = \phi^{l+kj}$ be the numerator in (11) and d be the denominator. If l is even then, $d = \phi^{l-1} + \cdots + \phi^7 + \phi^5 + \phi^3 + \phi$. If it is odd, then $d = \phi^{l-1} + \cdots + \phi^6 + \phi^4 + 2 \cdot \phi$. In the even case d is a finite number in base-ϕ form! In the odd case it is in extended base-ϕ form, which can be converted to base-ϕ (see the section on arithmetic in base-ϕ). Either way, the denominator has a finite base-ϕ representation, so by our previous result $d \in \mathbb{Q}(\sqrt{5})$. In addition, clearly $u \in \mathbb{Q}(\sqrt{5})$. Since $d \neq 0$, by the definition of a field $\frac{u}{d} \in \mathbb{Q}(\sqrt{5})$. This tells us that the infinite sum in (10) converges to an element $\mathbb{Q}(\sqrt{5})$, and therefore $q \in \mathbb{Q}(\sqrt{5})$.

($\Leftarrow$) Let $q \in \mathbb{Q}(\sqrt{5})$. Then,

$$q = \frac{a}{b} + \frac{c\sqrt{5}}{d} = \frac{ad + bc\sqrt{5}}{bd}, \quad a, b, c, d \in \mathbb{Z}.$$

Clearly, ad and bc are integers. Furthermore since $\sqrt{5} = 10.1_\phi$, $m = ad + bc\sqrt{5}$, $n = bd$ have finite base-ϕ representations (addition and multiplication by finite base-ϕ numbers always produces finite base-ϕ numbers; for the specific algorithms see the previous section). Since m and n have finite base-ϕ representations we can divide them by long division as outlined in the previous section. If n divides m then the quotient is in finite base-ϕ form. If n does not divide m evenly, then eventually the long division will be acting to the right of the decimal point in the dividend and thus selecting from only a finite number of remainders (which must eventually repeat) resulting in an infinite repeating representation. $\square$

Conclusion

We have shown how an irrationally based number system can be derived from the properties of ϕ itself. This golden ratio base provides an interesting study into how numbers are represented. This is especially true concerning so-called "rationals" and "irrationals"; we must consider the number as an abstract quantity and not merely in terms of their familiar base-ten representation. Algebraic number theory allows us to view these objects in terms of their relation to each other, and from it we can derive such intriguing results as seen in this report.

References

[1] Bergman, G. 1957. A Base Number system with an Irrational Base. Mathematics Magazine 31, (2): 98-110

[2] Rousseau, C. 1995. The Phi Number System Revisited. Mathematics Magazine 48, (4): 283-284

[3] Eggan, L.C. and Vanden Eynden, C.L. 1966. "Decimal" expansions to nonintegral bases. Am. Math. Monthly (73): 576-582